ESPRC Workshop Scaling in Social Systems
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# Scaling Laws for Cities

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## My Key Topics

- What Is Scaling? The Scaling Properties of Cities
- Space – Distance, Size – Frequency
- Seven Laws of Urban Scaling
- Three Exemplars
  - a) Allometry and Agglomeration
  - b) Size Distributions
  - c) Gravitational Interactions
- A First Attempt at Integrating Size, Scale & Interaction
- Open Questions: Defining Size and Choosing Scale
- Using Space, Scale & Size to Define Spatial Complexity

## What Is Scaling? The Scaling Properties of Cities

I am going to assume that you all know what scaling is but I will nevertheless introduce a simple explanation of these ideas as it is so central to complexity.

Scale is central to the way we organise our knowledge about cities – as we define them hierarchically from the region, even the nation state, down to the neighbourhood and even below to distinct communities clustered around streets.

The range of scale is thus bounded – and this means that we must be wary of models that presume change over all scales.



The biggest cities for example are in the order of millions – 25m Tokyo, Mexico City – and the smallest probably are in the order of hundreds, so the range is no more than 5 orders of magnitude.

If we look at physical form in cities, this scaling is reflected in fractal structures, statistically self-similar forms that repeat themselves across these scales with again no more than 5 orders of magnitude.

The same is more or less true with respect to the hierarchy of road systems, then.

Long before ideas about scaling and fractals surfaced, urbanists recognised this scaling in structure.

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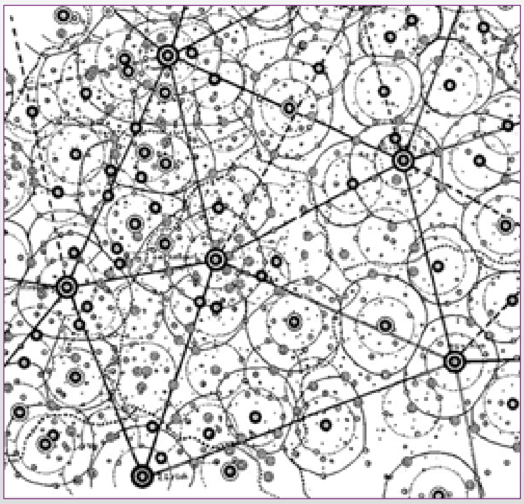
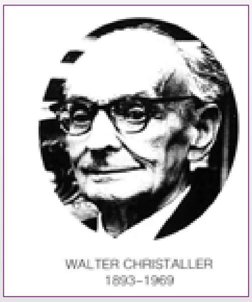
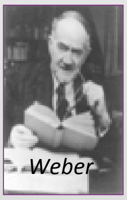
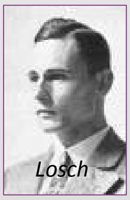
Let me break the ice and show some pictures of this before I become a little more abstract

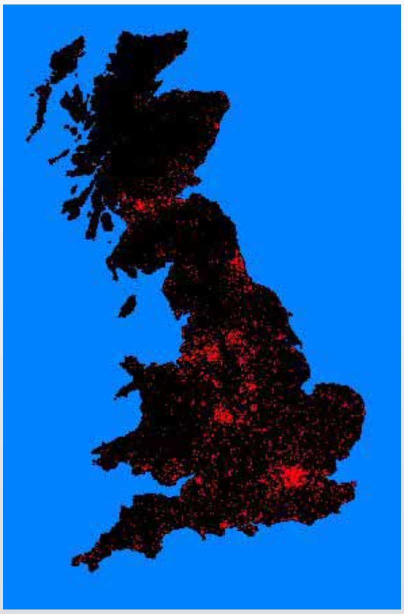


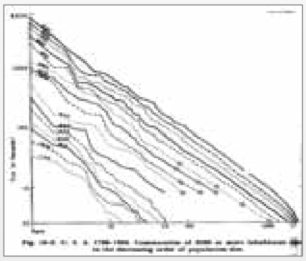






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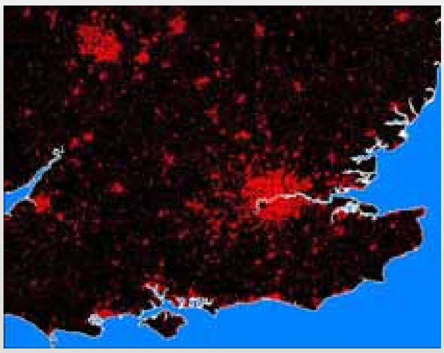

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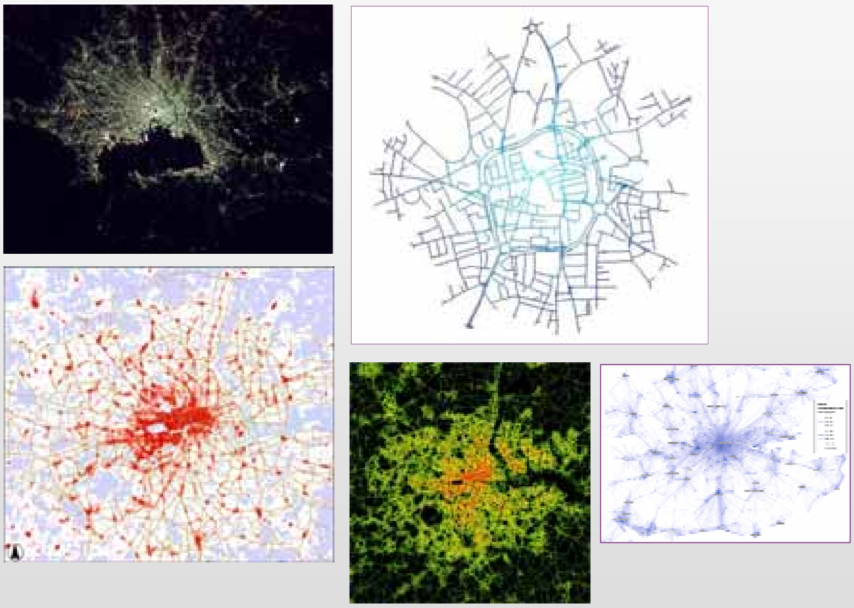

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**Space – Distance, Size – Frequency**

The scaling properties of cities are reflected in these morphologies and let me try to list them

a) Cities change shape as they change in size – this is allometry

b) There are many more small cities than big cities, and this scaling reflects competition for resources: to be a big city you must be a little city first

c) Cities are distributed with respect to their size in such a way that little cities are nested in the hinterlands of bigger cities. This implies big cities are spaced more widely than little cities – this is central place theory.

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d) People interact with each other more intensely in bigger than smaller cities. This is due to the fact that the no. of potential interactions in a population $P$ is $P^2$. In fact Dunbar's number suggests than the number of potential interactions has an upper bound of about 250 but the pressure to interact is greater in bigger cities.

e) People interact with one another less with increasing distance between them: this is the gravitational law.

f) Other kinds of interaction that diffuse over space, fall off with distance from their source. This tends to reduce the potential interaction effects of bigger cities.



I could go on informally with this list of properties and examine more detailed activities which comprise cities but the point I am making is that many things scale with the geometry of cities.

And there are many aspects to this geometry. So it follows there are many aspects to this scaling.

As far as I am aware, there is no good discussion yet of these scaling relations and the way they interact.

It would be nice to think that someone might produce a decent synthesis of these ideas but currently they are entangled with one another in ways that are hard to unravel. Let me summarise possible laws.

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**The Laws of Urban Scaling**

Let me try and formalise a little more how these scaling laws might be and have been developed. A word of warning. They may not be laws in the accepted sense of the term in the physical sciences but they are regularities that seem to persist in time and space.

*All others things being equal, ceteris paribus…….we can state the following about cities*

- As they grow, the number of 'potential connections' increases as the square of the population (*Metcalfe's Law*, the network equivalent of *Moore's Law*)
- As they grow, the average time to travel increases



- As they grow, the 'density' in their central cores tends to increase and in their peripheries to fall
- As they grow, more people travel by public transport
- As they get bigger, their average real income (and wealth) increases (the *Bettencourt-West Law*) – this is allometry.
- As they get bigger, they get 'greener' (*Brand's Law*)
- As they get bigger, there are less of them (*Zipf's Law*) – this is city size – rank size

Let me look briefly at the third of these observations: that is, as cities grow, the density in their central cores tends to increase and in their peripheries to fall

In fact in urban economics, there is a long tradition of generating monocentric city models where rents scale inversely with distance (or travel cost) from the core. This is probably *von Thunen's Law* after the German Count who first observed this on his estate in Saxony in 1826.

As a power law, this is central to spatial interaction, so we really need to a law of scaling that says that densities and rents decline as a power law with distance from their cores. In fact I am going to call this *Alonso's Law* after his setting the field alight in the early 1960s in resurrecting Alonso and applying the theory to cities.



**Three Exemplars**

*a) Allometry and Agglomeration*

As cities get larger, as they grow, they change in shape. Strictly speaking, no one has quite measured such qualitative change in terms of morphology as yet but a proxy for this is in terms of how their 'attributes' change with respect to population size.

An example relates income or wages or some measure of wealth $Y$ to population $P$ as

$$Y = KP^{\alpha}$$

where $K$ is a constant of proportionality and $\alpha$ is the scaling parameter

The scaling parameter can be greater than 1 $\alpha > 1$ which is positive allometry, less than 1 $\alpha < 1$, negative allometry, or equal to 1 $\alpha = 1$which is isometry.

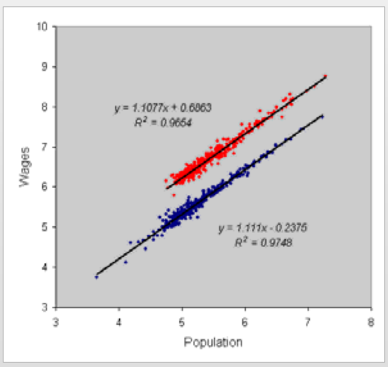
Greater than 1 is referred to as superlinearity and less than 1 sublinearity.

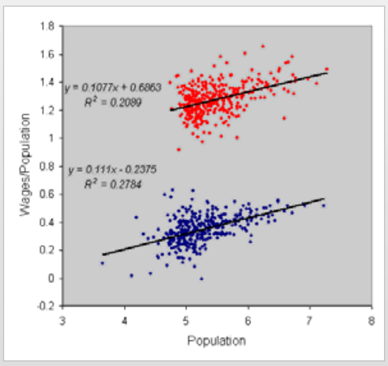
Bettencourt and West from Santa Fe have done most in this in recent years, particularly following West's work in biology on scaling and allometry. Much of this discussion is now about how big cities might be more wealthy, greener, more efficient and more divided can be predicated in these terms.

But let me illustrate with an example from the US data.



I, following the Santa Fe cities group, used the US Bureau of Economic Analysis on SMSAs from which I simply took their 366 regions for which population and income/wages data available from 1969 to 2008. For the first and last years in this data set



This shows distinct superlinearity, distinct economies of scale, notwithstanding a debate which is beginning as to how strong these relationships are.

This work produces extremely plausible evidence that the things that scale sublinearly in cities tend to be physical objects such as infrastructures, while things that scale superlinearly are attributes of populations that are highly specialist.

As we will see a little later, there are considerable problems in wrestling with the data for these kinds of problems and probably the way the national space economy has developed is significant in this.



*b) Size Distributions*

Our next relation involves the frequency $f(P)$ of different city sizes $P$ and this of course is Zipf's Law which we state as

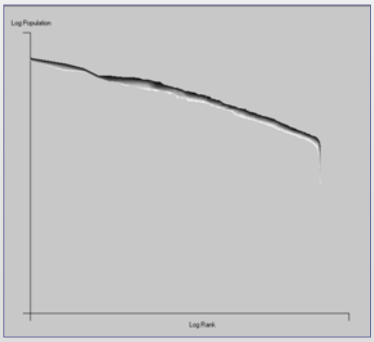
$$f(P) = KP^{-\beta}$$

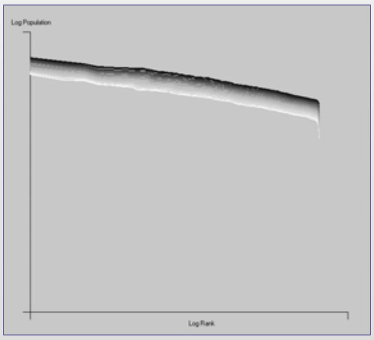
$K$ is still a constant of proportionality and $\beta$ is now the scaling parameter. Zipf's Law is usually presented in its counter cumulative form as the rank size rule and this can be stated from

$$F(P) = KP^{-\beta+1} = KP^{-\lambda}$$

We get the strict form of Zipf's Law when $\beta = 2$ , hence $\lambda = 1$

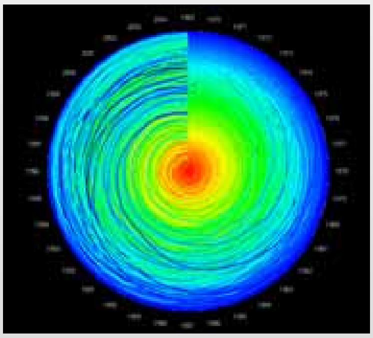
There are many illustrations of rank size that I might give but keeping to the SMSA data set used for exploring the allometry previously, then the ordering of population size and size of wages by rank is as follows

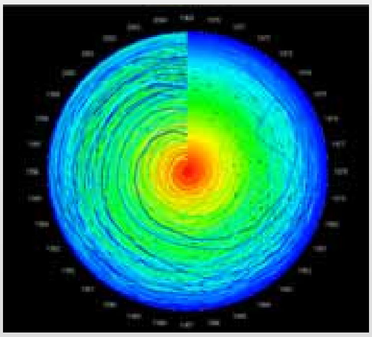


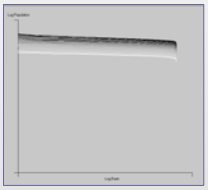


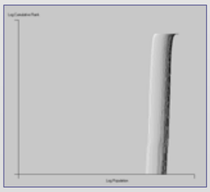

These are extremely regular and show little qualitative change over time. The volatility of the cities composing these relations is also quite muted compared to longer times, that is the cities do not change their ranks that much; they do a bit though

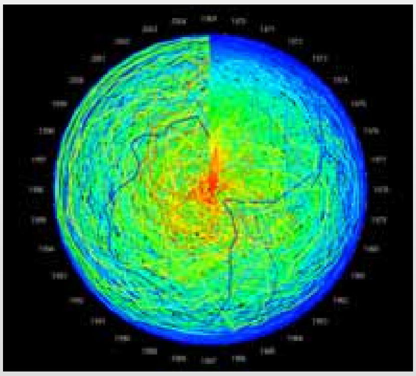




However when we look at their ratio, that is wages per capita, the volatility of this change is considerable. And represents a further puzzle. This however is a digression that I don't want to take any further here for my purpose is not to explore changes in rank









*c) Gravitational Interactions*

The last law that has been widely applied involves the analogy with gravitation that pertains to the interaction $T(P_i, P_j)$ between two populations which we can state as

$$T(P_i, P_j) = K \frac{P_i P_j}{d_{ij}^{\phi}}$$

where $d_{ij}$ is some deterrence, intervening opportunity or often distance and $\phi$ is the scaling parameter

Sometimes $\phi = 2$, the inverse square law, but often as in Zipf's Law, it is different from its theoretical equivalent value.

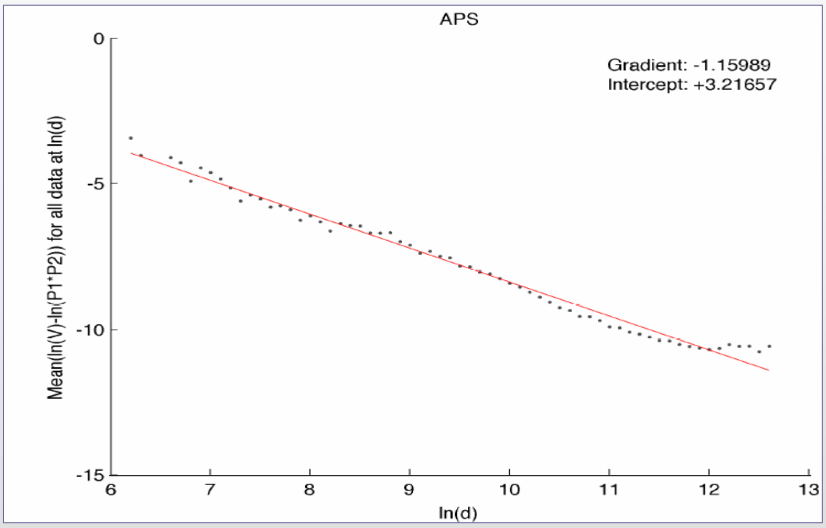
I should also say that our gravity model can be generalised to a population density model is we consider only one origin – such as the centre of the city and many destinations. Then we get the following sort of model which is very widely used and whose initial statement was by Colin Clark in 1951.

$$\rho_j = \frac{T(P_0, P_j)}{P_0 P_j} = K d_{ij}^{\phi}$$

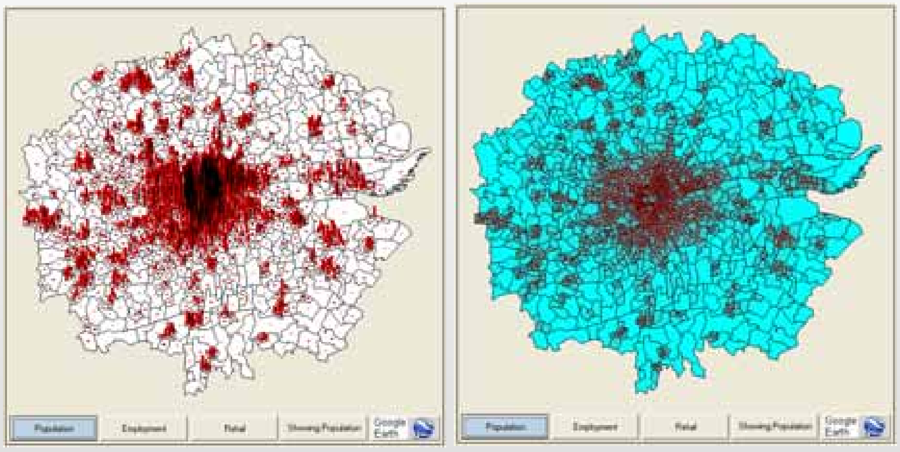
And this can also be generalised to a kind of rank-size where rank is unit distance from the CBD

There are many many illustrations of these kinds of distance decay relations.






And for then for wider region, we can see this as



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There are some trenchant debates surrounding these three types of relationship

Whether they are negative exponential, stretched exponential, or inverse power has preoccupied us a lot and substantial effort has put into ways in which power laws can be generated using simple models –

The heritage in this area is long and distinguished from Pareto, Yule, Lotka, Simon to Gabaix and Sornette et al. fusing urban growth theory with random stochastic models in the Gibrat tradition,

And in terms of allometry from Huxley, Haldane and so on through to the Santa Fe group.

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In spatial interaction from von Thunen to Alonso to Wilson etc. Last but not least, much of this work on scaling and self-similarity came out of the quantitative revolution in geography from the mid 1950s onwards from Garrison and Berry to Tobler, Getis, Nysteun, to Woldenberg and many others. I offer a glimpse of this world in Berry's 1964 paper




**A First Attempt at Integrating Size, Scale and Interaction**

We can define locations that relate to one another in terms of how populations relate. Locations intensify as people demand to be together to exchange in markets and it is usual for there to be a limited number of points where this takes place.

The density around these points is highest and the population then distributes itself around such points usually following some sort of inverse distance law as implied by urban density scaling.

Assume that everyone interacts with a market C. Then the distance from a point j to the market is $d_j$ and we assume the density $\rho_j$ follows an inverse square law– a power law (often a negative exponential) –

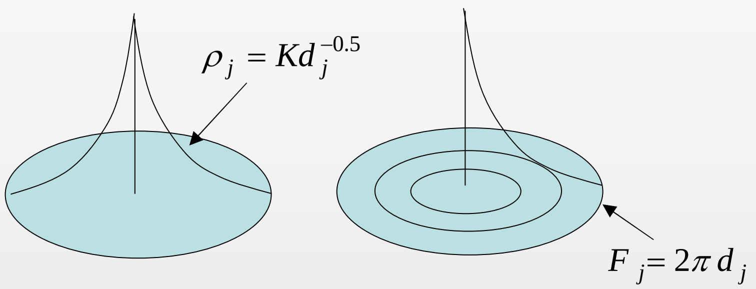
$$\rho_j = Kd_j^{-2}$$

Now we can plot a density cone in familiar form around the market centre C and note also that the number of points where people live around C varies according to the circumference of the circle at distance d from the centre, i.e. the no of locations is

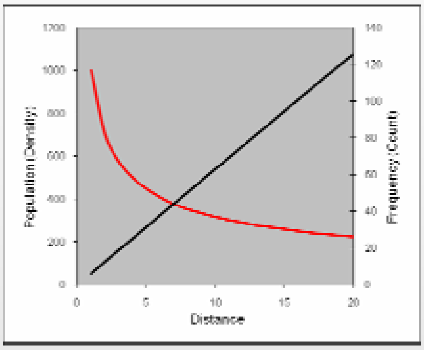
$$F_j = 2\pi d_j$$

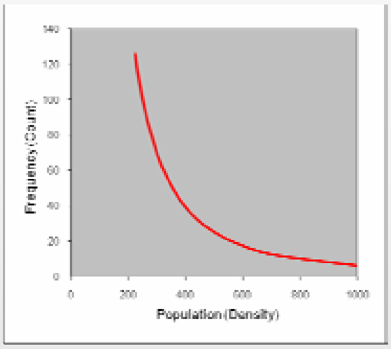
The size of each point is the density $\rho_j = Kd_j^{-2} = P_j$






We have changed the power of distance to 0.5 because this gives us a better result and is arguably relevant because we are really doing this on a line not on a space. Now let us see if this satisfies our basic scaling relations – let us count the frequency of different locations and compare these against different sizes.

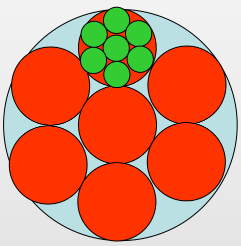



We thus compare $F_j$ against $\rho_j$ or $P_j$ for a simple numerical example. later. It is easy to show that the relation is dead simple and is (by construction) a power law, that is $F_j = GP_j^{-2}$ which leads directly to the rank size rule. Note the fact that distance could be unit rank – see Aura and Peter's work later



– albeit using a power of distance very different from the inverse square law. So we have establish that this functional form leads to scaling.



Now we can do exactly the same kind of exercise for a large space divided into a hierarchy of central places – we can assume a radius around the largest centre and calculate the total population, and then for successively smaller centres with smaller hinter-lands, we produce populations and then compare these against areas

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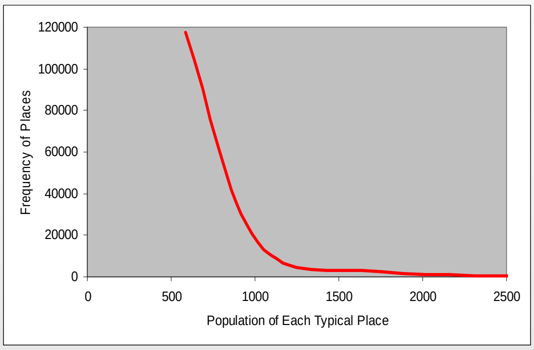
which are frequencies and which generate the same kind of rule. Our lattice is as shown above, and we can forget the spaces in between – Applying the same logic as for each circular town at each level and computing total populations in the hierarchy, we derive the same sort of scaling as follows.

First we assume a maximum radius d=1000 for the biggest all embracing central place – the blue circle and this gives the following total population as the integral of the density up to d=1000; the population is approximately 15811



Then at the next level down we divide the area of the largest circle into say, 7 red sub-circles each with radius 1000/3 and each of gives a population of 9128. We then get 49 areas at the next level down – the green circles each with a population of 5270 and so on, down to where we fix the lowest level at 40,353,607 circular areas, each with a population of 113.

If we then graph the frequency of this hierarchy against typical population size and plot the following graph which is clearly scaling.




This relation is linear on a log-log scale with the power of the size around 0.28, dependent of course on the assumptions we have made about how many levels of hierarchy there are and how each successive size is subdivided.

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Allometry of course relates to how these population scale with other phenomena and almost trivially we have defined them to scale with distance from C: and more roads are needed the smaller the populations are in this sort of monocentric city which is the sort of sublinearly we see in real cities. The equation is something like this

$$D^{0.5} = \int d(x)^{-0.5} dx = Z \int P(x) dx = P$$

In terms of income, we have not extended the model to income so we can say nothing about this.

This is the gist of an argument that might relate these various scaling laws.

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**Open Questions: Defining Size and Choosing Scale**

I will finish this rather general sweep through scaling and cities with a couple of open questions which I think are absolutely central to empirical work in this area.

To deciding what kinds of relationships we have and even to deciding whether there is the kind of regularity that we suggest for city systems

These are both to do with how we define our systems of interests first in terms of whether or not we have a ‘complete’ set of such objects, and second, whether or not we have the ‘right size’ of object.

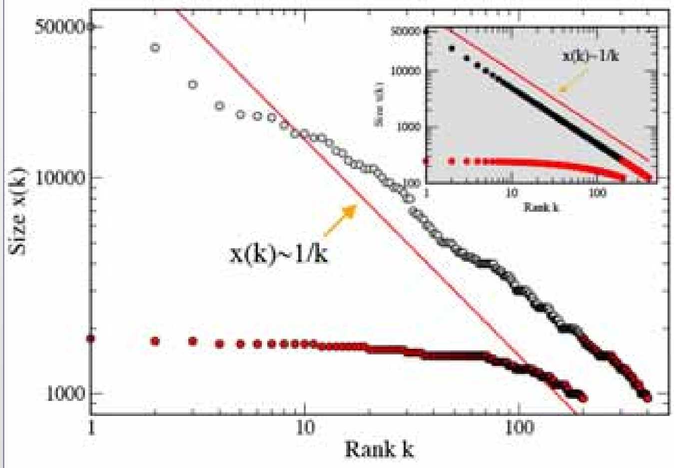


First then whether or not we have a complete set of objects and I will illustrate this for the rank size rule.

If we have a pure rank size distribution and we then omit the first half of the set of ranks – let us say 1 to n/2 in a set of n objects, and we then reorder that ranks of the set n/2 to n as a new set 1 to n/2, then this second set no longer follows a rank size distribution.

The distribution no longer has the coherence of the initial set.

Here we can show this for the following data which is the 390 US billionaires from the Forbes List in 2010.






There is a very simple message here. If we miss out any objects in our set where we know or assume scaling, then we will never be able to demonstrate scaling.

It is particularly crucial for city size distributions (and firm size too) because we often do not have decent control over how we pick our cities.

But more to the point, we often have to use cities (or firms) that pertain to national boundaries and we may want to examine size distributions that cross boundaries. This is a veritable minefield of problems for if we go the other way and merge two sets which are Zipfian, we do not get Zipf's Law

My second example pertains to how we define cities – in terms of their extent. This has been a major problem from time immemorial in that

a) the concept of a city has changed through time

b) Merging of cities into one another complicates the picture – Geddes' the father of town planning in Britain at least defined the term conurbation for this kind of polycentric structure

c) But in an urbanised world, where do towns begin and end

d) And last but not least, in a global world, cities merge into one another virtually or rather parts of cities do



We are engaged on testing the Bettencourt-West ideas on the UK – attempting to look at the question of proportionality and scaling in terms of physical and socio-economic attributes of towns of different size and our results are confusing.

Our best bet to date is the clustering of towns we have done at different levels and this follows the Hernan Makse work that I was involved in early on and has recently been extended by him, Gabaix and others.

What we will do is show the way we relax the threshold for including areas in a town using density and defining cores based on high densities in the first instance.

## Here is the clustering

I hope it will work as an animated gif – I am using an older powerpoint and it may open in explorer or quick time.

And here is are some results. I will ask Elsa to explain.

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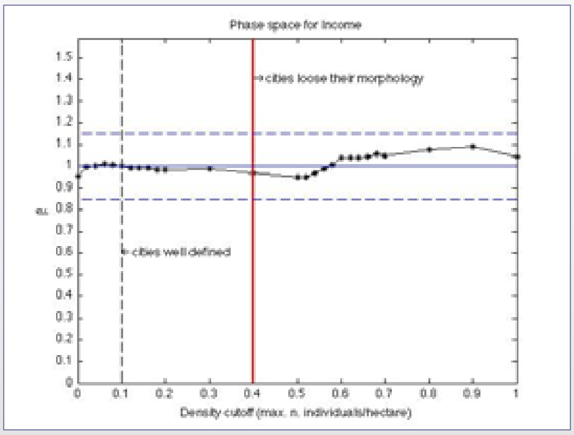


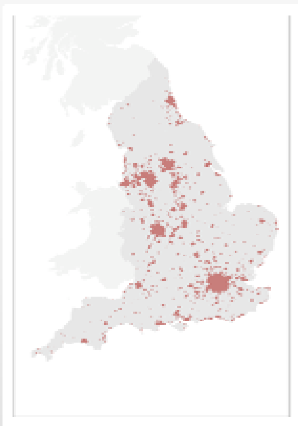

This is how the scaling parameter on population for income varies with different degrees of clustering. Note that the most appropriate clusters shown are at d=0.14

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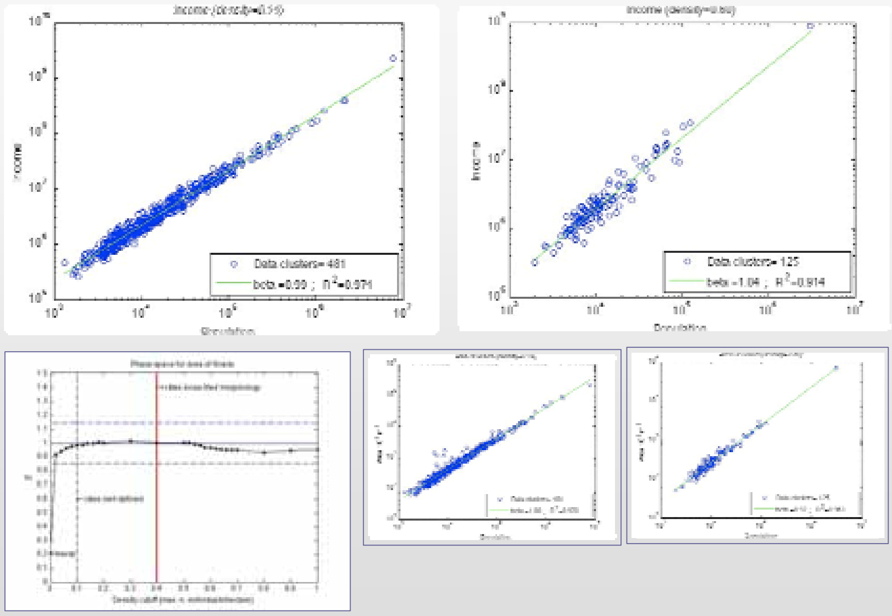




**Using Space, Scale & Size to Define Urban Complexity**

Last but not least we are also working on ways of trying to measure the complexity of these systems by using entropy formulas which merge space and scale – shapes of distributions with numbers of objects that define the distribution- using various formulas such as spatial entropy that build around Shannon

$$S = -\sum_i p_i \log \rho_i = -\sum_i p_i \log \frac{p_i}{\Delta x_i}$$

$$= -\sum_i p_i \log p_i + \sum_i p_i \log \Delta x_i$$

*This is the area size effect*

*This is the distribution and the number size effect in terms of n in entropy*

I am not going to conclude in any more depth than this as I am well over time but I hope some of these ideas will resonate with the talks during the rest of the meeting and generate some good discussion.

*I will post a version of the pdf of this powerpoint on the Presentation Pages of my web site. I gave a version in Oxford at a meeting on scaling last Thursday but this variant today is that posted with the Oxford title*

*http://www.complexcity.info/*

*Thanks*