TWENTY-NINTH NORTH AMERICAN MEETINGS OF THE REGIONAL SCIENCE ASSOCIATION

LINEAR URBAN MODELS

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ABSTRACT A class of linear models is developed in which activities are derived from transformations of each other and exogenous activities. The models are illustrated using spatial distributions of population and employment. Reduced forms are derived and the influence of different transformations on spatial model solutions is explored in terms of the balance of exogenous and endogenous variables, and through analysis of eigenstructures. Ten model types including the traditional Lowry model and Coleman's model of social exchange, are applied to an eight zone representation of Melbourne and the analysis is used to show how model solutions can be spatially independent of their inputs.

1. INTRODUCTION

Urban models are frequently characterised as being predominantly structured in linear or nonlinear terms, but in several contemporary developments both linear and nonlinear styles of analysis are intermixed. In spatial interaction modelling for example, models are usually derived through nonlinear optimisation leading to nonlinear model forms while such models are coupled together to form more general structures by means of linear accounting, subject to linear constraints. Thus these models can be analysed using both linear and nonlinear analysis, each type of analysis emphasising different properties of the model structure.

The traditional Lowry model is the classic example. The model was first stated by Lowry (1964) as an implicitly nonlinear structure. It was then developed in linear terms by Garin (1966) and Harris (1966) using analogies with input-output models, and then in nonlinear terms by Wilson (1974) who emphasised the derivation of its spatial interaction components through entropy-maximising. More recently, attention has been directed at coupling and solving the model's spatial interaction components in a more general nonlinear optimisation framework in which the model's linear structure is implicitly represented through its constraints (Wilson, Coelho, Macgill and Williams, 1981).

Most recent work has, in fact, been directed to the nonlinear analysis of such models. It is perhaps surprising that so little work has been concerned with exploring the models' linear structure, especially as such models appear more structurally transparent, and easier to extend and adapt in linear terms. Moreover, linear models have been developed extensively in regional science but hitherto there have been few attempts to generalise this class of models to explore their common properties. It is the purpose of this paper to present such a generalisation: to explore the effect of model structure on performance in terms of the balance of input and output variables and to clarify the question of choice of an appropriate model structure. These ideas will be illustrated using linear urban models in which activities are spatially distributed, although the ideas are also applicable to input-output, social exchange and various demographic accounts-based models.

The general model structure pertaining to this class is first stated and then adapted to two activities, the spatial distributions of population and employment.

Reduced forms are derived and various model types dependent upon different spatial distributional assumptions and input variables are characterised, including the Lowry model (Lowry 1964; Batty 1976) and Coleman's (1973) model of collective action based on the theory of social exchange. The effect of different distributional assumptions is then explored using eigenstructure analysis which gives a fairly comprehensive picture of the degree to which spatial solutions to these models are determined by model structure, input data and particular transformations. Applications to an eight zone model of Melbourne then serve to give these findings some empirical credibility. This suggests that much more research is required into invariance characterising spatial model solutions, the choice of inputs and outputs in particular model applications, and the level of resolution appropriate to any application. Although these issues pertain to the linear urban models discussed here, they are of more general import, and by way of conclusion, certain rules of thumb for good model design are presented.

2. A LINEAR FRAMEWORK FOR URBAN MODELS

Generalised Forms

To introduce the framework, first consider an activity y_1 distributed over n zones or sectors, and two activities y_2 and x_2 distributed over m zones or sectors. If y_2 and x_2 are lxm row vectors, y_1 is a lxn row vector and A_1 is an nxm matrix which transforms y_1 into y_2, then the linear model can be written as

$$y_2 = y_1 A_1 + x_2. \tag{1}$$

In a similar manner to Equation (1), it is possible to develop a chain of relationships in which y_z can be predicted as the sum of a transformation of y_{z-1} and exogenous variables x_z. For all variables y_z to be predicted however, the chain must be closed; that is at some point, $y_z = y_1$. The simplest possible case of Equation (1) is $y_2 = y_1$ which implies that $m = n$, and in this case Equation (1) could represent the structure of a conventional input-output model. In this context, it is necessary to examine the more general case where $z \geqslant 2$, and thus a suitable example of the closed sequence involves another equation for y_1 given as

$$y_1 = y_2 A_2 + x_1, \tag{2}$$

where x_1 is a lxn row vector and A_2 an mxn transformation matrix.

Solutions to the system of equations in (1) and (2) are given by the following reduced forms which result from substituting Equations (1) into (2) and (2) into (1). These are

$$y_1 = y_1 A_1A_2 + x_2 A_2 + x_1 \tag{3}$$

$$y_2 = y_2 A_2A_1 + x_1 A_1 + x_2. \tag{4}$$

In one sense, Equations (3) and (4) might be considered duals of one another. Explicit solutions for y_1 and y_2 can be given by rearranging (3) and (4), but at this stage, it is more appropriate to consider their solution in matrix split or iterative form as

$$y_1(t) = y_1(0)(A_1A_2)^t + (x_2A_2 + x_1) \sum_{r=0}^{t-1} (A_1A_2)^r \tag{5}$$

$$y_2(t) = y_2(0)(A_2A_1)^t + (x_1A_1 + x_2) \sum_{r=0}^{t-1} (A_2A_1)^r. \tag{6}$$

$y_1(0)$ and $y_2(0)$, $y_1(t)$ and $y_2(t)$ represent the starting and iteration t solution vectors to y_1 and y_2 respectively, while $(A_1A_2)^0$ and $(A_2A_1)^0$ are nxn, and mxm identity matrices.

Whether or not Equations (5) and (6) converge to unique vectors y_1 and y_2 depends upon the properties of A_1 and A_2. In this form, these iterative solutions refer to the static equilibrium framework based on Equations (1) to (4). However, Equations (5) and (6) could refer to dynamic versions of (3) and (4) with fixed inputs, and thus the framework developed here could easily be extended to cover dynamic models, for example the manpower and educational planning models of the type discussed by Bartholomew (1982). Furthermore, if there are no inputs to these models, that is if $\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{x}_2 = \mathbf{0}$, Equations (5) and (6) are similar to those of a first order process which if A_1 and A_2 were stochastic matrices would be a Markov process, equivalent to that developed by Coleman (1973). These possibilities will be explored further in the sequel.

An Urban Model of Activity Allocation-Distribution

To develop this framework further, it is necessary to make specific assumptions about the types of distribution and transformation involved. Two urban activities, employment e_i, $i = 1, 2, \ldots, n$ and population p_j, $j = 1, 2, \ldots, m$ are defined in spatial distributional terms so that

$$\sum_i e_i = 1 \text{ and } \sum_j p_j = 1,$$

and these are related through

$$e_i = \beta a_i + (1 - \beta)b_i, \qquad 0 \leq \beta \leq 1. \tag{7}$$

a_i is service and b_i basic employment in i, normalised so that

$$\sum_i a_i = \sum_i b_i = 1.$$

β is the ratio of service to total employment in the system.

Population is also considered as the sum of two components, internal population g_j and external (or basic) population h_j, which are defined so that

$$\sum_j g_j = \sum_j h_j = 1.$$

Population is then given by

$$p_j = \psi g_j + (1 - \psi)h_j, \qquad 0 \leq \psi \leq 1, \tag{8}$$

where ψ is the ratio of internal to total population.

It is assumed that basic employment and external population are exogenous variables equivalent to $\mathbf{x}_1$ and $\mathbf{x}_2$ defined earlier, and that service employment and internal population are endogenous; service employment is modelled as a linear function of population, and internal population as a linear function of employment defined respectively as

$$a_k = \sum_j p_j B_{jk}, \qquad \sum_k B_{jk} = 1, \tag{9}$$

and

$$g_j = \sum_i e_i A_{ij}, \qquad \sum_j A_{ij} = 1. \tag{10}$$

(B_{jk}) and (A_{ij}) are stochastic matrices which measure the demand by the population in zone j for services in zone k, and the demand by employees in zone i for housing in zone j respectively. These transformations are consistent with well-established ideas concerning spatial interaction (Wilson 1974) and it is assumed that each spatial transformation matrix is strongly-connected in the graph or network-theoretic sense.

Equations (7) and (8) can now be written in linked form. Substituting for a_i in (7) from (9), and g_j in (8) from (10) leads to

$$e_k = \beta \sum_j p_j B_{jk} + (1 - \beta) b_k$$

and

$$p_j = \psi \sum_i e_i A_{ij} + (1 - \psi) h_j.$$

In matrix terms, these equations can be written as

$$\boldsymbol{e} = \beta \boldsymbol{p}\, \boldsymbol{B} + (1 - \beta)\, \boldsymbol{b} \tag{11}$$

and

$$\boldsymbol{p} = \psi \boldsymbol{e}\, \boldsymbol{A} + (1 - \psi)\, \boldsymbol{h}. \tag{12}$$

Comparing Equations (11) with (2), and (12) with (1), it is clear that $\boldsymbol{e} = \boldsymbol{y}_1$, $\boldsymbol{p} = \boldsymbol{y}_2$, $\beta \boldsymbol{B} = \boldsymbol{A}_2$, $\psi \boldsymbol{A} = \boldsymbol{A}_1$, $(1 - \beta)\boldsymbol{b} = \boldsymbol{x}_1$, and $(1 - \psi)\boldsymbol{h} = \boldsymbol{x}_2$; thus the reduced forms in (3) and (4) and the matrix iterative solutions in (5) and (6) apply. It is, however, possible to say more about solutions to this model for the properties of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ have now been specified.

Solutions of the Urban Model

By analogy to Equation (3), the reduced form for employment in Equation (11) is given as

$$\boldsymbol{e} = \psi\beta \boldsymbol{e}\, \boldsymbol{A}\, \boldsymbol{B} + \beta(1 - \psi)\, \boldsymbol{h}\, \boldsymbol{B} + (1 - \beta)\boldsymbol{b}. \tag{13}$$

The three terms on the right hand side of (13) reflect service employment generated indirectly from basic employment and external population ($\psi\beta \boldsymbol{e}\, \boldsymbol{A}\, \boldsymbol{B}$), service employment generated directly from external population ($\beta(1 - \psi)\boldsymbol{h}\, \boldsymbol{B}$), and basic employment ($(1 - \beta)\, \boldsymbol{b}$). The ratios $\psi\beta$, $\beta(1 - \psi)$ and $(1 - \beta)$ also reflect the fractions of such employment in the model solution. In similar fashion, the reduced form of Equation (12) is given as

$$\boldsymbol{p} = \psi\beta\, \boldsymbol{p}\, \boldsymbol{B}\, \boldsymbol{A} + \psi(1 - \beta)\, \boldsymbol{b}\, \boldsymbol{A} + (1 - \psi)\boldsymbol{h}, \tag{14}$$

where $\psi\beta \boldsymbol{p}\, \boldsymbol{B}\, \boldsymbol{A}$ is the component of population associated with service employment, $\psi(1 - \beta)\boldsymbol{b}\, \boldsymbol{A}$ is the population directly associated with basic employment, and $(1 - \psi)\boldsymbol{h}$ is external population. Note that the ratios $\psi\beta$, $\psi(1 - \beta)$ and $(1 - \psi)$ reflect the fractions of each of these components in the final solution.

The matrix iterative solutions of Equations (13) and (14) starting from arbitrary but normalised vectors $\boldsymbol{e}(0)$ and $\boldsymbol{p}(0)$ can be stated by analogy to Equations (5) and (6) as

$$\boldsymbol{e}(t) = (\psi\beta)^t \boldsymbol{e}(0)(\boldsymbol{A}\, \boldsymbol{B})^t + [\beta(1 - \psi)\boldsymbol{h}\, \boldsymbol{B} + (1 - \beta)\boldsymbol{b}] \sum_{\tau=0}^{t-1} (\psi\beta)^\tau\, (\boldsymbol{A}\, \boldsymbol{B})^\tau \tag{15}$$

and

$$p(t) = (\psi\beta)^t p(0)(B\,A)^t + [\psi(1-\beta)b\,A + (1-\psi)h] \sum_{r=0}^{t-1} (\psi\beta)^r (B\,A)^r. \quad (16)$$

First we assume that $0 < \psi\beta < 1$, in short that exogenous inputs exist and endogenous activity is generated from them, and then in the next section we explore the special case where all activities are endogenously determined, $\psi\beta = 1$. Thus for $0 < \psi\beta < 1$, $(\psi\beta)^t \rightarrow 0$ as $t \rightarrow \infty$, and thus the first terms on the right hand side of Equations (15) and (16) converge to zero; the solutions thus depend on the second terms involving the matrix summations. As $A\,B$ and $B\,A$ are stochastic matrices, powers of these matrices are also stochastic, so convergence of these summations depends on $\psi\beta$.

We remarked earlier that equations such as (15) and (16) can be regarded as duals of one another; thus it is only necessary to illustrate results for one of them, for the other can be determined from the equilibrium relations in Equations (11) or (12). In the rest of this paper, we will only consider solutions to the population equation, Equation (16). Then in the limit,

$$p = \lim_{t\rightarrow\infty} p(t) = \lim_{t\rightarrow\infty} [\psi(1-\beta)b\,A + (1-\psi)h] \sum_{r=0}^{t-1} (\psi\beta)^r (B\,A)^r, \quad (17)$$

and as $(\psi\beta)^t (B\,A)^t \rightarrow \mathbf{0}$ as $t \rightarrow \infty$, the summation term in (17) is a converging geometric matrix series. It is easily shown that

$$p = \lim_{t\rightarrow\infty} p(t) = [\psi\,(1-\beta)b\,A + (1-\psi)h]\,[I - \psi\beta\,B\,A]^{-1} \quad (18)$$

which of course could have been derived directly from Equation (14). Using Equation (18) in (11) leads to

$$e = \beta[\psi(1-\beta)b\,A + (1-\psi)h]\,[I - \psi\beta\,B\,A]^{-1}\,B + (1-\beta)b,$$

and a dual form for p and e exists by taking e from the convergence of Equation (15) and substituting this into Equation (12).

3. THE ELABORATION OF LINEAR MODEL STRUCTURES

The Markov Model

The results given in Equations (7) to (8) relate to the case in which each activity is determined partly by some exogenous activity and partly as a transformation of another endogenous activity. In the case of this spatial urban model, basic employment might be considered as export orientated employment in the traditional economic sense, or as employment whose location it is impossible to model (Massey 1973). External population might reflect the same — either population dependent economically on activity outside the region or that whose location it is not possible to simulate, such as population located by some public agency. It is, however, instructive to examine three other variants of this model which reflect different balances of exogenous activity, thus different model structures.

First there is the case in which there is no external population, that is where $\psi = 1$. The resulting model is in effect one in which employment and population are now a function only of basic employment, and the model in this form is the conventional Lowry model as can easily be seen by making the appropriate simplifications to Equations (11) to (18). The model in fact is the Lowry model in its matrix form (Harris 1966; Garin 1966). Second, there is the case in which there is no basic employment, that is where $\beta = 1$. In this case, external population is the driving force of the model and this, it might be argued, is more appropriate for a model of a British New Town situation, say, where the population is located in a planned fashion. This model is the basic population equivalent of the Lowry model, and already the advantages of this general framework are becoming apparent in enabling the logic of linear urban models such as the Lowry model to be extended to other types of basic spatial determinant. In the sequel, these two single exogenous input models together with the model based on both inputs will be developed, but another interesting case of much greater analytic value emerges from the model with no exogenous inputs, that is where all population and employment are determined endogenously, where $\psi\beta = 1$.

It is worth examining this case in more detail. Equation (16) now becomes

$$p(t) = p(0)\,(B\,A)^t \tag{19}$$

and the solution depends on the behaviour of $(B\,A)^t$. $B\,A$ is a stochastic matrix and assuming again that $B\,A$ is strongly-connected which is an essential assumption of the spatial interaction transformation in any case, $(B\,A)^t$ converges to an idempotent matrix Z in which each row is identical. Then

$$\bar{p} = \lim_{t \to \infty} p(t) = p(0)(B\,A)^t = p(0)Z. \tag{20}$$

From Equation (20) it is clear that $\bar{p}$, the steady state population distribution, is equivalent to each identical row of the steady state matrix Z. As Z is idempotent, multiplication of (20) by $(B\,A)$ leads to

$$\bar{p} = \bar{p}\,B\,A,$$

which is equivalent to Equation (14) with $\psi\beta = 1$. Clearly the steady state population distribution is the steady state of a discrete Markov process. In similar fashion, it can be shown that $(A\,B)^t$ converges to an idempotent matrix Q as $t \to \infty$ and using the same logic as above, the steady state employment $\bar{e}$ is given as

$$\bar{e} = \bar{e}\,A\,B.$$

Furthermore, when $n = m$, $Q = Z\,B$ and $Z = Q\,A$; in short, when $\psi\beta = 1$, Equations (15) and (16) represent dual Markov processes.

In this form, the model is equivalent to Coleman's (1973) model of collective action in which the equilibrium can be interpreted as the outcome of an exchange process. In fact, an exchange interpretation could quite easily be developed for urban models such as these in the same spirit as that developed by Coleman, thus enabling insights into these types of models to be enriched further (Batty 1981). Moreover the framework developed here shows how the Coleman model might also be seen as a special case of a more general model of collective action in which such action is seen as being determined by both endogenous and exogenous factors. However, such interpretations are beyond the immediate concern of this paper.

The Steady State Model

The model without exogenous inputs, referred to hereafter as the Coleman model, although interesting in its own right as a distinct model structure in the framework, is also useful in that it highlights the fact that the transformation matrix $(B\,A)^t$ converges towards the idempotent matrix Z as the number of iterations of the process of solution increases. As any stochastic row vector multiplied by this idempotent matrix gives a row of this matrix, this implies that in the case where the transformation matrix is or becomes idempotent, the exogenous vector then has no influence on the resulting solution. This is the condition of invariant distributional regularity identified by Schinnar (1978).

To demonstrate this idea, consider the case where the matrix $\boldsymbol{B}\,\boldsymbol{A}$ is already idempotent, that is

$$\boldsymbol{B}\,\boldsymbol{A} = (\boldsymbol{B}\,\boldsymbol{A})^t = \boldsymbol{Z},\ t > 0.$$

Then for the case where $0 < \psi\beta < 1$, the matrix series in Equation (17) can be written as

$$\begin{aligned} \lim_{t\to\infty} \sum_{r=0}^{t-1} (\psi\beta)^r\,(\boldsymbol{B}\,\boldsymbol{A})^r &= \boldsymbol{I} + \boldsymbol{Z} \sum_{r=1}^{\infty} (\psi\beta)^r, \\ &= \boldsymbol{I} + \psi\beta(1 - \psi\beta)^{-1}\,\boldsymbol{Z}. \end{aligned} \tag{21}$$

Using Equation (21) in (17), the equilibrium population referred to as the population from the steady state model, now becomes

$$\hat{\boldsymbol{p}} = [\psi(1 - \beta)\boldsymbol{b}\,\boldsymbol{A} + (1 - \psi)\boldsymbol{h}]\,[\boldsymbol{I} + \psi\beta\,(1 - \psi\beta)^{-1}\boldsymbol{Z}],$$

which simplifies, using the fact that each row of $\boldsymbol{Z}$ is $\bar{\boldsymbol{p}}$, to

$$\hat{\boldsymbol{p}} = \psi\beta\,\bar{\boldsymbol{p}} + \psi(1 - \beta)\boldsymbol{b}\,\boldsymbol{A} + (1 - \psi)\boldsymbol{h}. \tag{22}$$

Equation (22) shows that population $\hat{\boldsymbol{p}}$ is a function of the input data, and of the steady state $\bar{\boldsymbol{p}}$ and this implies that the input has no influence on the endogenously generated population.

A similar result holds for employment. Substituting Equation (22) into (11) gives

$$\hat{\boldsymbol{e}} = \psi\beta\,\boldsymbol{b}\,\boldsymbol{A}\,\boldsymbol{B} + \beta(1 - \psi)\,\boldsymbol{h}\,\boldsymbol{B} + (1 - \beta)\boldsymbol{b}.$$

Now as $\boldsymbol{B}\,\boldsymbol{A} = \boldsymbol{Z}$, and $\boldsymbol{A}\,\boldsymbol{B}\,\boldsymbol{A} = \boldsymbol{A}\,\boldsymbol{Z} = \boldsymbol{Z} = \boldsymbol{Q}\,\boldsymbol{A}$, then $\boldsymbol{A}\,\boldsymbol{B} = \boldsymbol{Q}$, and the steady state employment can now be written as

$$\hat{\boldsymbol{e}} = \psi\beta\bar{\boldsymbol{e}} + \beta(1 - \psi)\boldsymbol{h}\,\boldsymbol{B} + (1 - \beta)\,\boldsymbol{b}, \tag{23}$$

which has the same structure as Equation (22). From Equations (22) and (23), it is clear that the degree to which $\hat{\boldsymbol{p}}$ approaches $\bar{\boldsymbol{p}}$ and $\hat{\boldsymbol{e}}$ approaches $\bar{\boldsymbol{e}}$ depends on the ratio $\psi\beta$. Three types of model based on Equations (22) and (23), namely $0 < \psi < 1$ and $0 < \beta < 1$; $\psi = 1$; and $\beta = 1$, are developed in the applications presented below.

Invariant Distribution and Idempotence in Transformation

There are several different ways in which the matrices $\boldsymbol{B}\,\boldsymbol{A}$ and $\boldsymbol{A}\,\boldsymbol{B}$ may be idempotent. It is clear that if $\boldsymbol{A}$ or $\boldsymbol{B}$ is idempotent, then either of their products is idempotent. Then if $\boldsymbol{A}$ is idempotent, that is if the probability of residing in any place is independent of the place of employment, Equation (22) simplifies to

$$\hat{\boldsymbol{p}} = \psi\bar{\boldsymbol{p}} + (1 - \psi)\boldsymbol{h},$$

while if $\boldsymbol{B}$ is idempotent, that is if the probability of demanding services in any place is independent of the place where that demand is generated, Equation (23) simplifies to

$$\hat{\boldsymbol{e}} = \beta\bar{\boldsymbol{e}} + (1 - \beta)\boldsymbol{b}.$$

If both A and B are idempotent, then both Equations (22) and (23) simplify in the manner shown.

Another possible effect of idempotence which will be developed in the sequel involves the situation where

$$B\,A = (B\,A)^t = I,\ t > 0.$$

In this situation, Equation (17) simplifies to

$$\bar{p} = \frac{\psi(1-\beta)}{(1-\psi\beta)}\,b\,A + \frac{(1-\psi)}{(1-\psi\beta)}\,h, \tag{24}$$

and using Equation (24) in (11) gives

$$\bar{e} = \frac{\beta(1-\psi)}{(1-\psi\beta)}\,h\,B + \frac{(1-\beta)}{(1-\psi\beta)}\,b. \tag{25}$$

In this case, the equilibrium population and employment distributions are simply proportional to the appropriately scaled fraction of each exogenous distribution of activities. Clearly this situation can arise in several ways. For example if both B and A are identity matrices where $m = n$ then this implies no spatial interaction in the system whatsoever; that is employees live and work in the same zone and demand their services there. The same situation can also arise if the patterns of spatial demand for housing are the inverse of those for services, that is where $A = B^{-1}$ and $B = A^{-1}$. In all these cases, $B\,A = I$ and $A\,B = I$, but these patterns and the resulting identical locational distributions can clearly arise under very different conditions of spatial interaction.

In the applications which follow in a later section, three types of model will be developed: the model based on actual interaction matrices A and B given in Equations (13) and (14), the model based on the steady state interaction patterns derived from $Z = \lim_{t\to\infty} (B\,A)^t$ and $Q = \lim_{t\to\infty} (A\,B)^t$ in Equations (22) and (23), and the model based on Equations (24) and (25) in which it is assumed that this pattern is associated with no interaction or self cancelling interaction. For each of these models, three model structures will be tested; first where both inputs are present ($0 < \psi < 1$ and $0 < \beta < 1$), second where basic employment is the only input ($\psi = 1$), and third where external population is the only input ($\beta = 1$). Finally the Markov model with no inputs, where $\psi\beta = 1$, Coleman's model, will be developed, thus giving 10 different models in all to be explored.

4. THE MEASUREMENT OF DISTRIBUTIONAL INVARIANCE

Spectral Decomposition of a Stochastic Matrix

In the previous section, we indicated that in the absence of input data, the ultimate distribution of activities in the model would depend on their steady state distribution matrices Z and Q. In the case where there are exogenous inputs and where the transformation matrices are already in the steady state, the solutions can be derived as a weighted sum of the steady state and input distributions. This suggests that in the case of the general model, it may be possible to measure formally the degree to which the distribution matrices $B\,A$ and $A\,B$ approach the steady state when solutions to the model are derived.

To demonstrate the relationship, we will use $B\,A$ and its steady state $Z =$

$\lim_{t\to\infty} (\boldsymbol{B}\ \boldsymbol{A})^t$. Assuming that the eigenvalues of $\boldsymbol{B}\ \boldsymbol{A}$ are all distinct, the matrix $\boldsymbol{B}\ \boldsymbol{A}$ can be represented as

$$\boldsymbol{B}\ \boldsymbol{A} = \boldsymbol{R}^T\ \Lambda\ \boldsymbol{S} = \sum_{j=1}^{m} \lambda_j r_j^T s_j = \sum_{j=1}^{m} \lambda_j \boldsymbol{V}_j, \tag{26}$$

where $\boldsymbol{R}$ is an mxm matrix of right-hand eigenvectors of $\boldsymbol{B}\ \boldsymbol{A}$, $[r_j]$, $\boldsymbol{S}$ is an mxm matrix of left-hand eigenvectors $[s_j]$, and Λ is an mxm diagonal matrix of the m eigenvalues of $\boldsymbol{B}\ \boldsymbol{A}$ where each eigenvalue λ_j on the diagonal Λ_{jj} is associated with the eigenvectors $\boldsymbol{r}_j$ and $\boldsymbol{s}_j$. $\boldsymbol{V}_j = r_j^T s_j$ and this matrix is defined as the spectral set. Assuming that the scales of $\boldsymbol{s}_j$ and r_j^T are chosen so that $s_j r_j^T = 1$, then $\boldsymbol{V}_j$ satisfies the relations

$$\boldsymbol{V}_\ell \boldsymbol{V}_j = \boldsymbol{0} \text{ if } \ell \neq j;\ \boldsymbol{V}_\ell \boldsymbol{V}_j = \mathrm{V}_j \text{ if } \ell = j; \text{ and } \sum_{j=1}^{m} \boldsymbol{V}_j = \boldsymbol{I}. \tag{27}$$

These results are taken from Bäiley (1964). (T indicates the matrix transpose operation.)

The decomposition defined in Equations (26) and (27) enables the powers of $\boldsymbol{B}\ \boldsymbol{A}$ to be expressed in simple form as

$$(\boldsymbol{B}\ \boldsymbol{A})^t = \sum_{j=1}^{m} \lambda_j^t\ \boldsymbol{V}_j. \tag{28}$$

From the Perron-Frobenius theorem (see Heal, Hughes and Tarling 1974), a stochastic matrix such as $\boldsymbol{B}\ \boldsymbol{A}$ has a dominant eigenvalue equal to 1, and all other eigenvalues of the matrix have an absolute value less than 1. Assuming these values are distinct (slight perturbation of the values in $\boldsymbol{B}\ \boldsymbol{A}$ will normally ensure this within an acceptable error bound for $\boldsymbol{B}\ \boldsymbol{A}$), then it is possible to order the eigenvalues and eigenvectors of $\boldsymbol{B}\ \boldsymbol{A}$ so that $\lambda_1(=1) > |\lambda_2| > |\lambda_3| > \ldots > |\lambda_m|$. In the case of the dominant eigenvalue $\lambda_1 = 1$,

$$\lambda_1^t\ \boldsymbol{V}_1 = \lambda_1 r_1^T s_1 = \boldsymbol{1}^T s_1 = \boldsymbol{Z}, \tag{29}$$

because the right-hand eigenvector associated with $\lambda_1 = 1$ must be the unit vector, that is $\boldsymbol{B}\ \boldsymbol{A}\ \boldsymbol{1}^T = \boldsymbol{1}^T$ and $\boldsymbol{s}_1$ represents the steady state vector associated with $\boldsymbol{B}\ \boldsymbol{A}$, that is $\boldsymbol{s}_1 = \boldsymbol{s}_1\ \boldsymbol{B}\ \boldsymbol{A}$, the left-hand eigenvector. Using Equation (29) in (28), it is possible to write the powers of $\boldsymbol{B}\ \boldsymbol{A}$ as

$$(\boldsymbol{B}\ \boldsymbol{A})^t = \boldsymbol{Z} + \sum_{j=2}^{m} \lambda_j^t\ \boldsymbol{V}_j, \tag{30}$$

where it is clear that in the limit as $t \to \infty$, $(\boldsymbol{B}\ \boldsymbol{A})^t \to \boldsymbol{Z}$, and $\sum_{j=2}^{m} \lambda_j^t\ \boldsymbol{V}_j \to \boldsymbol{0}$. Thus the difference between the matrix $\boldsymbol{B}\ \boldsymbol{A}$ and its steady state has the simple form

$$\boldsymbol{B}\ \boldsymbol{A} - \boldsymbol{Z} = \sum_{j=2}^{m} \lambda_j\ \boldsymbol{V}_j,$$

and this difference converges to $\boldsymbol{0}$ as $t \to \infty$.

Spectral Representation of a Linear Urban Model

It is now possible to represent the matrix series which arises in the iterative solution to the model in Equation (17) using (28) as

$$\sum_{r=0}^{\infty} (\psi\beta)^r (B\,A)^r = \sum_{r=0}^{\infty} \sum_{j=1}^{m} (\psi\beta\lambda_j)^r V_j. \tag{31}$$

Then as $|\lambda_j| \leq 1$ and $0 < \psi\beta < 1$, the series in (31) can be simplified as follows

$$\sum_{r=0}^{\infty} (\psi\beta\lambda_j)^r = (1 - \psi\beta\lambda_j)^{-1}$$

and

$$\sum_{r=1}^{\infty} (\psi\beta\lambda_j)^r = \psi\beta\lambda_j \sum_{r=0}^{\infty} (\psi\beta\lambda_j)^r = \psi\beta\lambda_j (1 - \psi\beta\lambda_j)^{-1}.$$

Equation (31) can now be written as

$$\begin{aligned}\sum_{r=0}^{\infty} (\psi\beta)^r (B\,A)^r &= \sum_{j=1}^{m} (1 - \psi\beta\lambda_j)^{-1} V_j, \\ &= I + \sum_{j=1}^{m} \psi\beta\lambda_j (1 - \psi\beta\lambda_j)^{-1} V_j, \\ &= I + \psi\beta(1 - \psi\beta)^{-1} Z + \sum_{j=2}^{m} \psi\beta\lambda_j(1 - \psi\beta\lambda_j)^{-1} V_j.\end{aligned} \tag{32}$$

Using Equation (32) in Equation (17) enables the general model to be written as

$$\begin{aligned}p = {} & [\psi(1 - \beta)b\,A + (1 - \psi)h][I + \psi\beta(1 - \psi\beta)^{-1} Z \\ & + \sum_{j=2}^{m} \psi\beta\lambda_j(1 - \psi\beta\lambda_j)^{-1} V_j],\end{aligned} \tag{33}$$

which using Equations (21) and (22) simplifies to

$$\begin{aligned}p &= \psi\beta\hat{p} + [\psi(1 - \beta)b\,A + (1 - \psi)h][I + \sum_{j=2}^{m} \psi\beta\lambda_j (1 - \psi\beta\lambda_j)^{-1} V_j] \\ &= \hat{p} + [\psi(1 - \beta)b\,A + (1 - \psi)h][\sum_{j=2}^{m} \psi\beta\lambda_j(1 - \psi\beta\lambda_j)^{-1} V_j].\end{aligned} \tag{34}$$

The second term on the second line of Equation (34) is the difference $p - \hat{p}$ and this is the percent deviation of p from the steady state distribution $\hat{p}$. Equations (33) and (34) represent a new decomposition of the traditional linear urban model, and the same logic can be easily transferred to any such model in which the transformation of one endogenous activity into another can be separated into a scale and distribution effect. This of course limits the usefulness of spectral decomposition for input-output analysis but it is highly relevant to urban models such as the Lowry model (Batty 1979). Equations for e analogous to (33) and (34) can also be derived and it is possible to derive dual relationships between the spectral sets of $B\,A$ and $A\,B$. As these are not of central relevance here, they will not be formally presented.

Distributional Differences between Model Types

The decomposition in Equation (34) can now be simplified as follows. First set the inputs and the deviation from the steady state matrix Z as

$$w = \psi(1 - \beta)b\,A + (1 - \psi)h$$

and

$$\Sigma = \sum_{j=2}^{m} \psi\beta\lambda_j(1 - \psi\beta\lambda_j)^{-1}V_j.$$

Equation (34) now becomes

$$p = \psi\beta\tilde{p} + w + w\,\Sigma, \tag{35}$$

which is referred to as the canonical form of the linear urban model. It is the sum of a steady state effect $\psi\beta\tilde{p}$, an input effect w of order $(1 - \psi\beta)$, and a deviation from the steady state through compounding of inputs $w\,\Sigma$.

Differences between the population distributions of the three models p, $\hat{p}$ and $\bar{p}$, as well as the Markov model $\tilde{p}$ can now be stated. Then $p - \tilde{p}$ from Equation (35) is given in terms of the three effects as

$$p - \tilde{p} = (\psi\beta - 1)\tilde{p} + w + w\,\Sigma,$$

which clearly sums to zero, as the first two terms are of order $\psi\beta - 1$ and $1 - \psi\beta$ which cancel, and $w\,\Sigma$ is a deviation. The difference $p - \hat{p}$ is only in terms of these deviations from the steady state

$$p - \hat{p} = w\,\Sigma,$$

while $p - \bar{p}$ can be given in terms of the three effects

$$p - \bar{p} = \psi\beta\tilde{p} + w\,\Sigma - \psi\beta(1 - \psi\beta)^{-1}\,w.$$

Other differences $\tilde{p} - \hat{p}$, $\tilde{p} - \bar{p}$ and $\hat{p} - \bar{p}$ are just functions of the steady state and the input data for the deviations $w\,\Sigma$ are only associated with the full model. Analogous relationships for e, $\hat{e}$, $\bar{e}$ and $\tilde{e}$ can be derived in dual form or as functions of the relationships given here.

5. APPLICATIONS

Comparison of Model Types

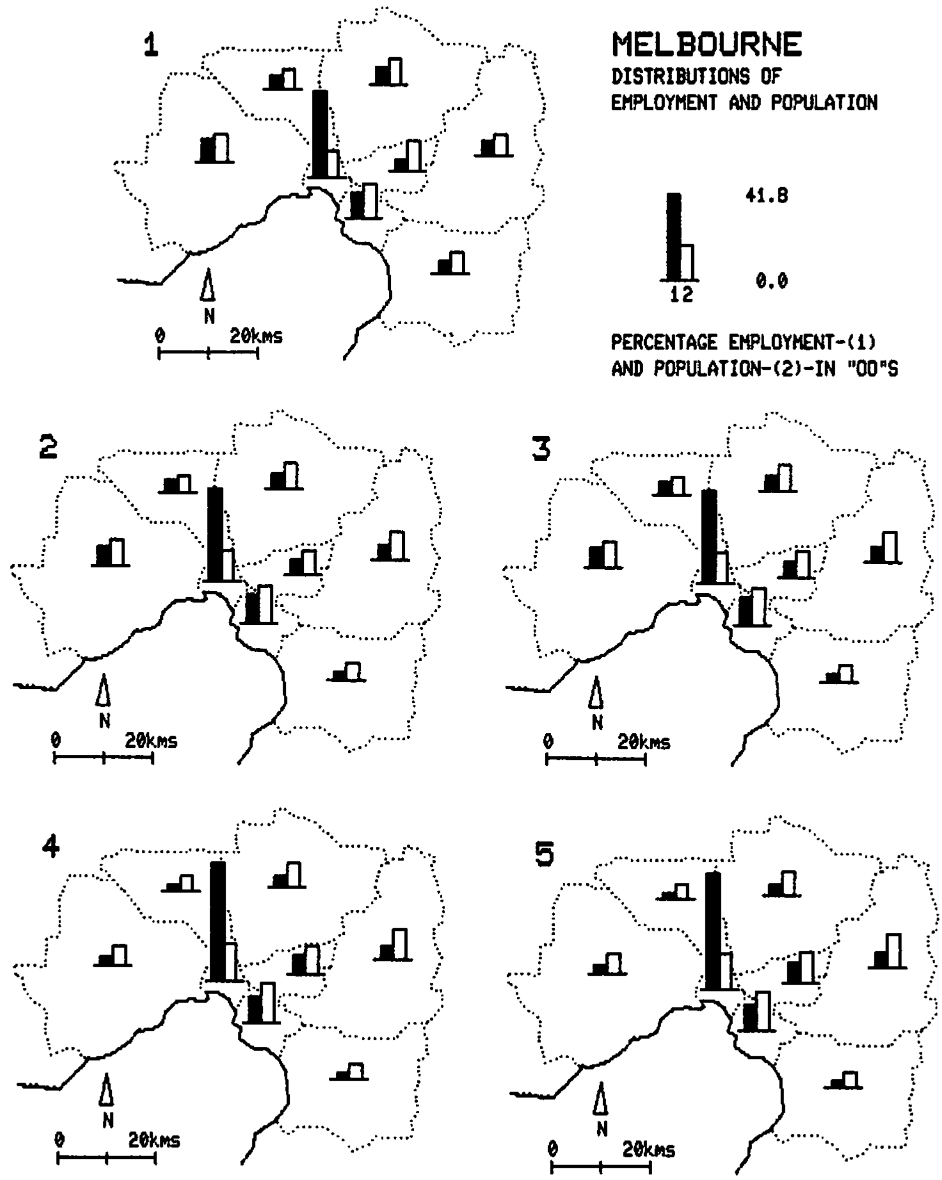
To demonstrate the degree of spatial invariance contained in the different model structures introduced above, these models have been applied to an eight zone representation of the Melbourne metropolitan region. In this case, the observed pattern of employment is highly concentrated in the CBD and surrounding zones while the distribution of population is much more evenly spread. Basic employment (employment in primary and manufacturing industries) is more evenly spread than total employment but is concentrated in the CBD and the west of the city. External population, measured as population in public housing, is considerably more concentrated than total population, in the CBD and in the west of the city like basic employment. In this case, $n = m = 8$ and the patterns of observed population and employment are shown in map 1 of Figure 1. The distribution of basic employment and external population are not shown separately but in fact their distribution is equivalent to employment in map 10 and population in map 11, both illustrated as part of Figure 3. This example, although at a coarse level of spatial resolution, is a reasonably realistic one in that it is typical of the differences in activity distribution charactertising many western cities, and models of these cities.

As there are 11 different distributions of population and employment to compare (from 10 model types together with the observed distributions), these have been arranged in the following order, using indices u, $v = 1, 2, \ldots, 11$ to represent the particular distribution in question. Index 1 refers to the observed distributions while indices 2, 3 and 4 refer to the full model based on actual interaction: 2 refers to the model with both inputs, 3 to that with only basic employment (the Lowry model) and 4 to that with only external population. Model 5 is that based on no exogenous inputs, that is the Markov or Coleman model. Indices 6, 7 and 8 refer to the steady state model in Equations (22) and (23); 6 is the full model with both inputs, 7 the model with only basic employment, and 8 the model with only external population. Finally, indices 9, 10 and 11 refer to the models based on 'no interaction,' given in Equations (24) and (25). Model 9 is the full model with both inputs, 10 with only basic employment and 11 with only external population. The maps 1-11 which are produced in Figures 1 to 3 refer to population and employment distributions from each of these model types. In the sequel, any value of population p_j or employment e_i will be superscripted by its model type index, u, v where necessary.

As a first step in evaluating and comparing the 10 models and the observed distributions, the ratios of endogenous to exogenous activity — population and employment — associated with each model are presented in Table 1. This table shows immediately the differences in model structure in terms of the presence or absence of inputs and outputs as well as the overall weight of exogenous and endogenous variables in determining the ultimate distribution of population and employment. From Table 1, it is clear that the 10 model types cover a wide range of assumptions concerning the effect of input and output variables, from models based entirely on input data — models 9, 10 and 11 to that based on no input data but only on the effect of the spatial transformations — model 5. Note also that model 10 predicts employment as entirely basic employment, and model 11 population as entirely external population.

It is also possible to speculate on similarities and differences between the models' predictions, from the prior assumptions embodied in Table 1. Employment and population have almost identical determinants in terms of the importance of inputs and outputs, and thus it is to be expected that similarities and differences between models will be consistent in terms of population or employment. Then there are the strong similarities between the actual interaction models (2, 3 and 4) and the steady state interaction models (6, 7 and 8), and differences between these will be entirely in terms of the differences of the transformation matrices from their steady states. Because external population is such a small fraction of total population, models based on this as the only input (models 4 and 8) are likely to be similar in their predictions to model 5, the Coleman model, which is based on no inputs. These models too are likely to be fairly different from the others, as will be the models based on no interaction in which the inputs entirely determine the predictions (that is, models 9 and 10 which are similar in themselves and model 11).

The predicted distributions of population and employment from these models are presented in map form in Figures 1 to 3. In Figure 1, the observed distributions and the four models u = 2, 3, 4, 5 (that is the model based on both inputs, the two models based on single inputs, and the Markov model based on no inputs) are presented. Models 2 and 3 are similar to each other and to the observed distributions, while models 4 and 5 give a much stronger concentration of employment in the CBD. However, the pattern of population generated by these models is close to the observed pattern. Figure 2 presents the three steady state models, models 6, 7 and 8, which on casual inspection appear close to their actual interaction equivalents, models 3, 4 and 5. This indicates that the effect of the transformation matrices is close to their steady state forms. Figure 3 presents three more extreme models, models 9,10 and 11 based on the 'no interaction' assumption in which population and employment are direct functions



1 OBSERVED EMPLOYMENT AND POPULATION DATA
2 BASIC EMPLOYMENT AND BASIC POPULATION MODEL:OBSERVED INTERACTION
3 BASIC EMPLOYMENT MODEL:OBSERVED INTERACTION
4 BASIC POPULATION MODEL:OBSERVED INTERACTION
5 STEADY STATE MARKOV MODEL WITH NO BASIC INPUTS:OBSERVED INTERACTION

FIGURE 1. Observed and Predicted Distributions of Employment and Population for the Models Based on Observed Interaction Patterns

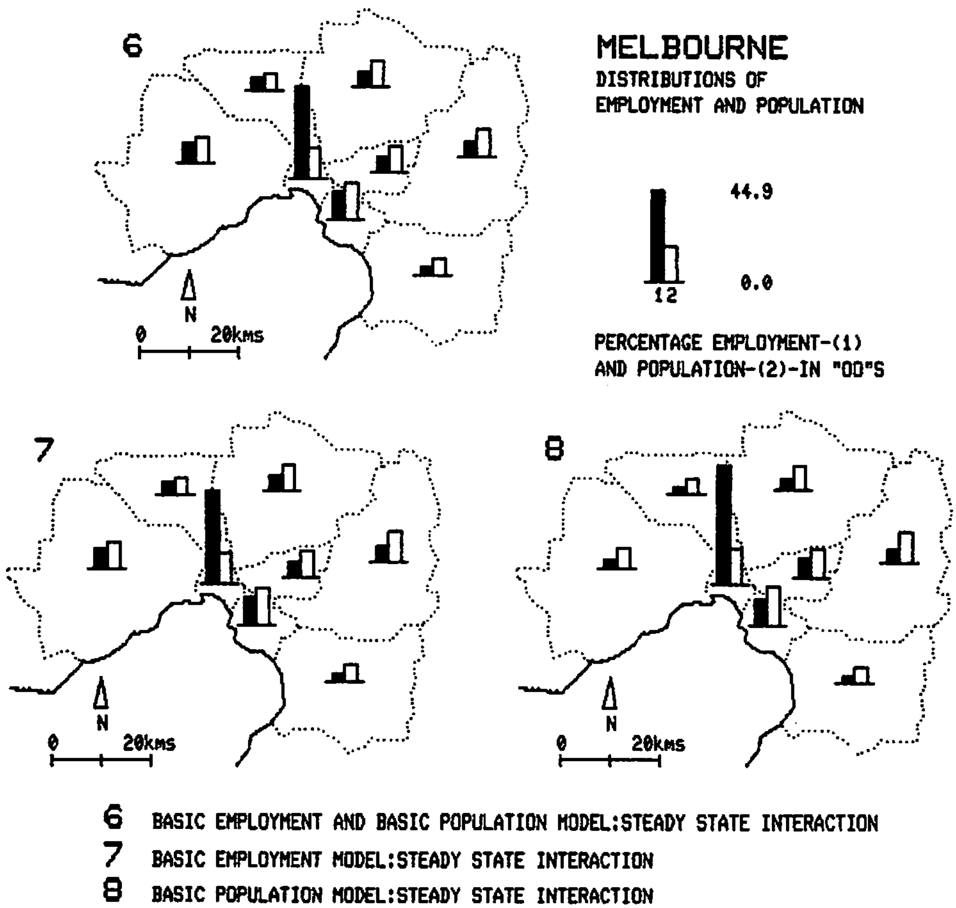


FIGURE 2. Predicted Distributions of Employment and Population for the Models Based on Steady State Interaction Patterns

of the associated input data. Models 9 and 10 generate distributions of total employment and population which are all close to the distribution of basic employment, while model 11 predicts much more concentrated distributions equivalent to the distribution of external population. Although we have referred to models 9, 10 and 11 as the 'no interaction' case, this is not, strictly speaking correct in that we are not assuming $A = B = I$. All we assume is that $\mathbf{B}\ \mathbf{A} = I$ and $A\ B = I$, situations which can arise in many ways. However, it is possible to see the cases of actual 'no interaction' for in these cases, population in model 10 would have an identical distribution to employment and employment in model 11 an identical distribution to population.

It remains to make more precise the casual comparisons emerging from Figures 1 to 3. Accordingly, we have computed percent differences between the various distributions for each pair of models. Then the percentage difference θ_{uv} between models u and v for population is given as

$$\theta_{uv} = \frac{100}{m} \sum_j \frac{|p_j^u - p_j^v|}{p_j^u},$$

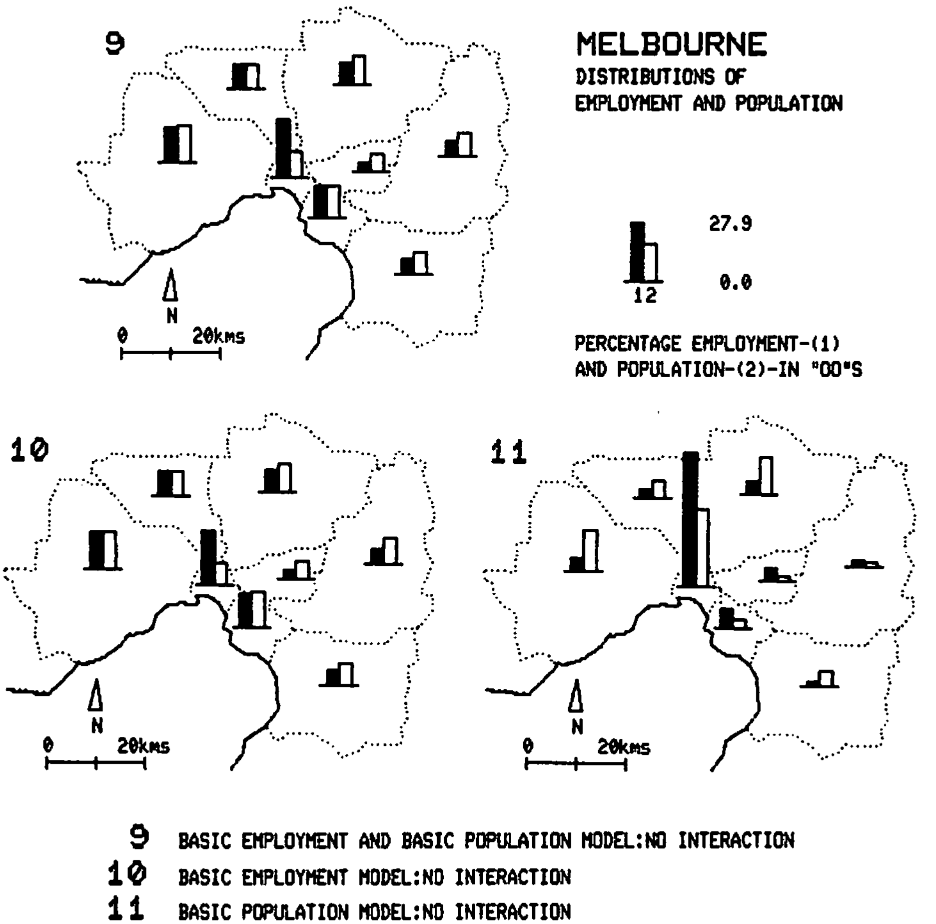


FIGURE 3. Predicted Distributions of Employment and Population for the Models Based on the 'No Interaction' Type Assumption

and the percent difference for employment θ_{uv} is given as

$$\phi_{uv} = \frac{100}{n} \sum_i \frac{|e_i^u - e_i^v|}{e_i^u}$$

These percentages are presented in Table 2 for the 11 distributions of population and employment respectively. These tables bear out previous observations. In terms of the observed situation, the models in which basic population is the sole determinant, and those which embody the 'no interaction' assumption perform least well. The Markov model is not close to the observed situation either but the steady state and actual interaction models where basic employment dominates, are close to the observed situation. This suggests that the basic employment input is a major determinant of a well-fitting model of this particular system, a point which will be made cogent in the next section. Table 2 contains a large quantity of comparative information, and read with Figures 1 to 3 provides a rich source for evaluating these various models which can be further developed

TABLE 1. Classification of Model Types by Weight of Variables

MODEL TYPES, u		POPULATION: Endogenous: Service Population	POPULATION: Exogenous: Basic Population	POPULATION: Exogenous: External Population	EMPLOYMENT: Endogenous: Service Employment	EMPLOYMENT: Exogenous: Basic Employment	EMPLOYMENT: Exogenous: External Employment
		$\psi\beta$	$\psi(1 - \beta)$	$(1 - \psi)$	$\psi\beta$	$(1 - \beta)$	$\beta(1 - \psi)$
2 ACTUAL INTERACTION	$0 < \psi\beta < 1$	0.6153	0.3506	0.0341	0.6153	0.3629	0.0217
3 ACTUAL INTERACTION	$\psi = 1$	0.6371	0.3629	0	0.6371	0.3629	0
4 ACTUAL INTERACTION	$\beta = 1$	0.9659	0	0.0341	0.9659	0	0.0341
5 MARKOV (Coleman)	$\psi\beta = 1$	1	0	0	1	0	0
6 STEADY STATE INTERACTION	$0 < \psi\beta < 1$	0.6153	0.3506	0.0341	0.6153	0.3629	0.0217
7 STEADY STATE INTERACTION	$\psi = 1$	0.6371	0.3629	0	0.6371	0.3629	0
8 STEADY STATE INTERACTION	$\beta = 1$	0.9659	0	0.0341	0.9659	0	0.0341
9 NO INTERACTION	$0 < \psi\beta < 1$	0	0.9114	0.0886	0	0.9436	0.0564
10 NO INTERACTION	$\psi = 1$	0	1	0	0	1	0
11 NO INTERACTION	$\beta = 1$	0	0	1	0	0	1

TABLE 2. Percentage Differences between Model Types

PERCENT DEVIATIONS IN POPULATION θ_{uv}

	v = 1	2	3	4	5	6	7	8	9	10	11
1	0	13	13	22	23	14	14	23	14	17	69
2	13	0	3	14	15	1	4	14	21	21	62
3	13	3	0	13	14	2	1	13	22	22	65
4	24	15	14	0	3	13	13	1	38	38	68
5	25	17	16	3	0	15	15	2	40	40	72
u = 6	14	1	2	12	14	0	2	13	22	22	63
7	14	4	1	12	13	2	0	12	23	23	66
8	24	15	14	1	2	14	13	0	38	38	69
9	17	21	22	34	36	22	23	35	0	6	64
10	19	22	22	35	37	23	23	36	7	0	76
11	156	156	162	178	186	159	165	179	127	139	0

PERCENT DEVIATIONS IN EMPLOYMENT ϕ_{uv}

	v = 1	2	3	4	5	6	7	8	9	10	11
1	0	13	13	39	39	15	15	39	30	34	42
2	14	0	1	30	30	2	3	30	44	49	38
3	14	1	0	30	30	2	2	30	44	49	38
4	71	51	51	0	2	48	48	0	125	133	33
5	72	51	51	2	0	49	49	1	127	134	35
u = 6	16	2	2	29	29	0	1	29	46	51	37
7	16	3	2	29	29	1	0	29	46	51	38
8	71	51	51	0	1	48	48	0	126	133	34
9	26	38	38	63	63	39	39	63	0	4	65
10	30	42	42	67	67	43	43	67	4	0	69
11	78	62	63	43	45	62	62	43	125	132	0

by the reader. Note, however, that the matrices in Table 2 are not symmetric for the base of comparison between any pair of models depends on the first model in the pair.

Spatial Invariance and the Effect of Model Structure

To take the analysis one stage further, it is worth exploring in quantitative terms how close the tranformation matrices are to their steady states, and how the predicted distributions vary as the balance between endogenous and exogenous activity changes. Comparing the predicted distributions of population and employment for the actual interaction models with their associated steady state interaction counterparts reveals extremely small percentage differences; that is, for population and employment respectively in models 2 and 6, these are 1.48 and 2.35; for models 3 and 7, these are 1.20 and 1.37; and for models 4 and 8, these are 0.69 and 0.43. We can compare these with the percentage difference between the matrix $\mathbf{B}\ \mathbf{A}$ and its steady state form $\mathbf{Z}$ given as

$$\rho = \frac{100}{m^2} \sum_{k\ell} \frac{\left| \sum_{j=2}^{m} \lambda_j \, V_{k\ell j} \right|}{Z_{k\ell}}.$$

In this example, ρ is 31.15 which is considerably different from the ultimate percentage differences between the locational distributions.

However, it is necessary to take account of the convergence of $\mathbf{B}\ \mathbf{A}$ towards $\mathbf{Z}$ for only a small fraction of the difference $\mathbf{B}\ \mathbf{A} - \mathbf{Z}$ will be transmitted to the ultimate distributions. Thus a more useful statistic is based on the percentage differences between the actual compounded effects of $\mathbf{B}\ \mathbf{A}$ given by $\{[\mathbf{I} - \psi\beta\ \mathbf{B}\ \mathbf{A}]^{-1} - \mathbf{I}\}$ and the steady state effects given by $\psi\beta(1 - \psi\beta)^{-1}\mathbf{Z}$. The statistic which is based on Equation (32) is given as

$$\Omega = \frac{100}{m^2} \sum_{k\ell} \frac{\left| \sum_{j=2}^{m} \psi\beta\lambda(1 - \psi\beta\lambda_j)^{-1} V_{k\ell j} \right|}{\psi\beta(1 - \psi\beta)^{-1} Z_{k\ell}}.$$

The value of Ω is 8.94 which implies that there is about a 9 percent difference between the actual spatial transformation of the exogenous inputs into their ultimate form and the transformation in the steady state form which is independent of such inputs. On aggregation of these differences to derive locational distributions, the percentage difference will thus be reduced to an order of 1 or 2 percent.

To complete this analysis it is worthwhile examining the same effect but excluding the actual spatial transformations contained in the eigenvectors of $\boldsymbol{BA}$. Then the ratio

$$\mu = 100 \frac{\sum_{j=2}^{m} \psi\beta\lambda_j(1 - \psi\beta\lambda_j)^{-1}}{\psi\beta(1 - \psi\beta)^{-1}},$$

gives the value of 31.14 which is close to the original percentage difference between the matrices $\boldsymbol{B}\,\boldsymbol{A}$ and $\boldsymbol{Z}$. In other words, it is the similarities between the components of $\boldsymbol{V}_j$ and $\boldsymbol{Z}$ rather than their strength which determines their effect. This can also be seen by examining the vector of eigenvalues of $\boldsymbol{B}\,\boldsymbol{A}$ given as

$$[\lambda_j] = [1.00, 0.26, 0.17, 0.11, 0.09, 0.04, 0.03, 0.02]$$

where the eigenvalues are all real and positive but the ratio

$$\sum_{j=2}^{m} \frac{\lambda_j}{\lambda_1}$$

is now of the order of 70 percent. However, it is easy to see that the eigenvalues λ_j, $j \neq 1$ converge quickly towards zero in the iterative solution to the model, and measures which link these values to particular stages of the solution have been used to measure convergence to the invariant solution (Batty 1979). Considerable research, however, remains to be done in developing this type of analysis in linear urban modelling, and this demonstration can only be regarded as a first attempt to explore the problem.

To complete this analysis, the full model given in Equations (13) and (14) has been solved for a series of values of ψ and β in the range 0 to 1. The population and employment vectors $\boldsymbol{p}$ and $\boldsymbol{e}$ from each solution have been compared with their observed distributions, with the appropriate steady state distributions $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{e}}$ computed for each set of values of ψ and β, and with the Markov solutions $\tilde{\boldsymbol{p}}$ and $\tilde{\boldsymbol{e}}$. Values of ψ and β at regular increments of 0.1 in the range 0 to 1 have been selected, thus giving 11 values of each ratio, a total of 121 varieties of each model to apply. For the actual interaction and steady state models, the extreme values of ψ and β, that is $\psi = 0, 1$: $\beta = 0, 1$ give the same solutions: when $\psi, \beta = 0$, $\boldsymbol{p} = \hat{\boldsymbol{p}} = \boldsymbol{h}$, $\boldsymbol{e} = \hat{\boldsymbol{e}} = \boldsymbol{b}$; when $\psi, \beta = 1$, $\boldsymbol{p} = \hat{\boldsymbol{p}} = \tilde{\boldsymbol{p}}$, $\boldsymbol{e} = \hat{\boldsymbol{e}} = \tilde{\boldsymbol{e}}$; when $\psi = 1$ and $\beta = 0$, $\boldsymbol{p} = \hat{\boldsymbol{p}} = \boldsymbol{b}\,\boldsymbol{A}$, $\boldsymbol{e} = \hat{\boldsymbol{e}} = \boldsymbol{b}$; and when $\psi = 0$, $\beta = 1$ $\boldsymbol{p} = \hat{\boldsymbol{p}} = \boldsymbol{h}$, $\boldsymbol{e} = \hat{\boldsymbol{e}} = \boldsymbol{h}\,\boldsymbol{B}$. Thus the range over which these models are solved includes the no interaction and Markov models developed in an earlier section.

Response surfaces plotted as contours of the percentage differences between various model solutions, observed distributions and steady state model types are illustrated in Figure 4. In Figures 4(a) and (b) these differences are shown in terms of the observed distributions and these surfaces indicate the set of values which provide a model with the closest fit to the observed distributions. For population, the best model is that with no external population and with the ratio of service to total employment about 0.4. This is close to that observed and suggests that the Lowry model is most suitable for this example. In terms of employment, however, the best model is that with $\psi \approx 0.7$ and $\beta \approx 0.5$. We have not taken this type of analysis any further but the notion of selecting the best type of model in terms of the balance of endogenous to exogenous variables is a further spinoff from developing a general framework such as this for linear urban models.

The substantial range of percentage differences between predicted and observed distributions (from 70 to 10 percent for population) is not repeated when the predictions based on actual and steady state interactions are compared in Figures 4(c) and (d). Here the greatest percent difference is only about 4 percent and this bears out the fact that the input assumptions are obviously more critical in the spatial variation in the model's solutions than the spatial transformations. Finally a comparison between model predictions and the steady state distributions from the Markov model are presented in Figures 4(e) and (f). Here the range of variation is quite substantial (from 0 to over 130 percent) and this once again indicates that the existence of inputs is a major determinant of spatial distribution. These points have some significance for model design.

6. CONCLUSIONS

One obvious rule in applying the urban models developed here to spatial distributions relates to partitioning such distributions so that endogenous and exogenous population and employment distributions are radically different. This, it has been argued, will ensure a meaningful spatial transformation of activities into one another. However, as shown here, if such transformations are close to their steady states, then it is the transformations rather than any input distribution which are significant. Furthermore, if the input activity is only a small fraction of the total and if the transformation is near the steady state, the model is close to the endogenous Markov version. In contrast if the input data is a large fraction and the transformations far from their steady states, the solutions will be quite sensitive to the influence of both these distributions. This suggests that there is no a priori set of rules which indicates how much or how different input distributions should be from one another, but that the overall weight, their distribution and the nature of their spatial transformation within the model should all be considered together to judge the quality and non-triviality of such spatial models.

The analysis introduced here, particularly that involving the spectral decomposition of spatial transformations, is very much an initial foray into the question of invariance in model solutions. This is part of a broader question concerned with the extent to which spatial distributions are 'averaged' on transformation, a question which hitherto has received little attention in spatial interaction modelling and which pertains to both linear and nonlinear model representations. In future work, such questions will be developed in greater detail with the ultimate intention of deriving statistics from spectral or variance analysis which will provide less ambiguous indicators of the importance of invariance than those used here. This, together with further applications of the models in this framework to nonspatial examples, represent the main lines for future research.

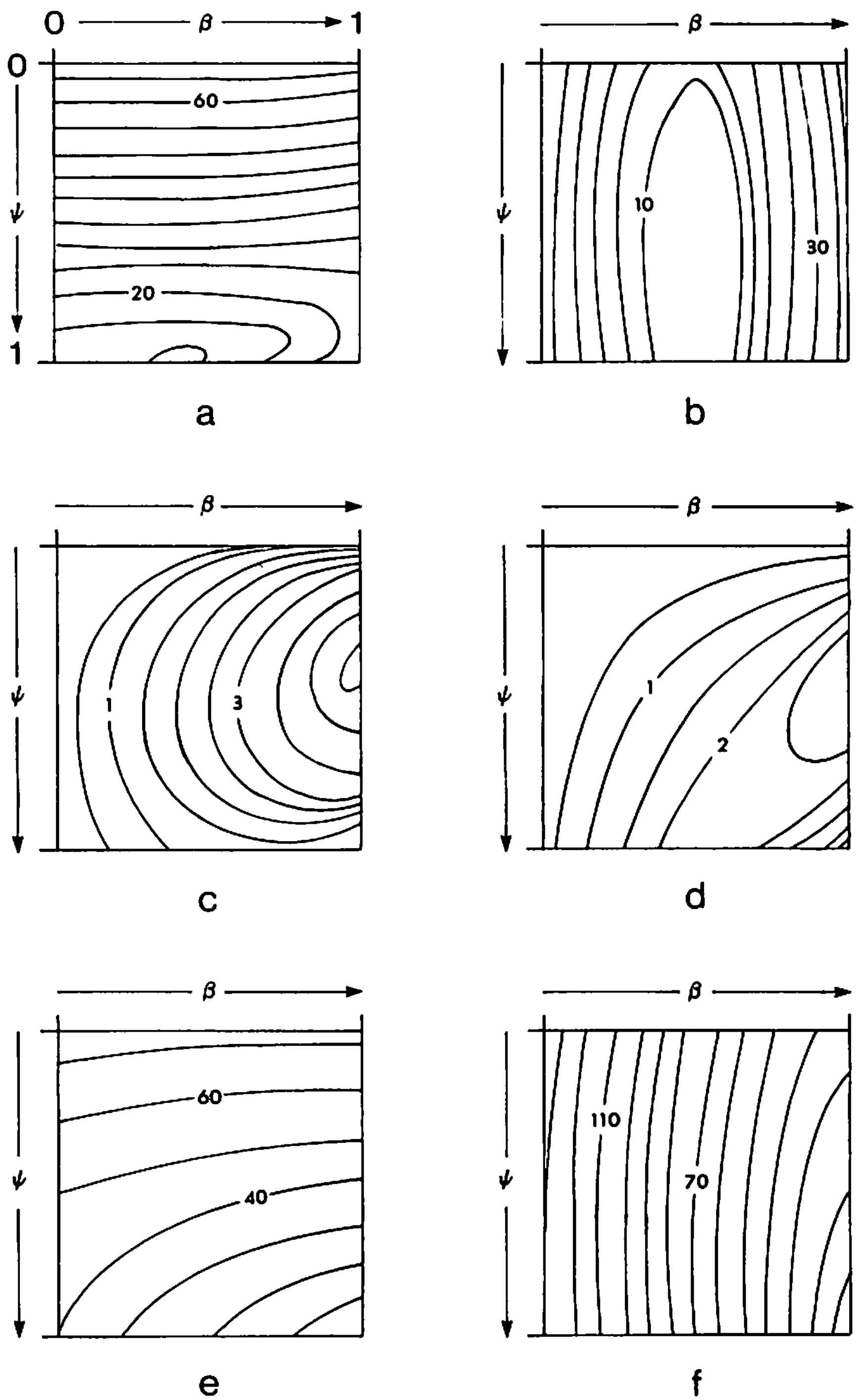


FIGURE 4. Comparisons of Model Types Over the Range of Assumptions Concerning Weight of Inputs

ACKNOWLEDGEMENTS

The research was carried out when the author was Visiting Fellow at the University of Melbourne. The author gratefully acknowledges the support of the University and advice concerning data for Melbourne from Dr. Ray Wyatt. The paper also forms part of contract HR 8046/1 of the Social Science Research Council, London.

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